Problem Sheet 4 for Supervision in Week 11

1. Recall the proof in the notes of the result that there do **not** exist integers x, y such that $15x^2 - 7y^2 = 1$. (See also PJE, q.1 on p.225.)

   Use similar ideas to show that

   i) There are no integral solutions to $30x^2 - 23y^2 = 1$,
   ii) There are no integral solutions to $5x^2 - 14y^2 = 1$,
   iii) ★ If $n \equiv 1 \bmod 5$ then there are no integers x and y such that $n = 30x^2 + 22y^2$.

   Hint: the idea is to choose a modulus m and then to look at the equation $\bmod m$. I suggest that in (i) and (ii) you choose m so that one of the terms in the equation vanishes $\bmod m$.

2. i) Show that there are no integral solutions to $2x^3 + 27y^4 = 23$. (Hint; look at this modulo 9.)
   ii) Show that there are no integral solutions to $7x^5 + 3y^4 = 4$.
   iii) ★ Show that 7 never divides $a^4 + a^2 + 2$ for $a \in \mathbb{Z}$.

3. Write out the multiplication tables for $(\mathbb{Z}_6, +)$ and $(\mathbb{Z}_6, \times)$.

4. ★ Write out the multiplication table for $(\mathbb{Z}_9^*, \times)$ and list the inverses of each element.

5. Find the inverses of each of the following:

   (i) $[2]_{93}$, (ii) $[5]_{93}$, (iii) $[25]_{93}$, (iv) $[32]_{93}$.

6. For each of the following relations on $\mathbb{N}$, list the ordered pairs that belong to the relation.

   (i) $\mathcal{R} = \{(x, y) : 2x + y = 9\}$,

   (ii) $\mathcal{S} = \{(x, y) : x + y < 7\}$,

   (iii) $\mathcal{T} = \{(x, y) : y = x^2\}$.

7. ★ For each of the following relations on the set $\{1, 2, 3, 4\}$, indicate whether it is reflexive, symmetric or transitive. **Give your reasons.**

   (i) $\mathcal{R}_1 = \{(2,2), (2,3), (2,4), (3,2), (3,3), (3,4)\}$,

   (ii) $\mathcal{R}_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$,

   (iii) $\mathcal{R}_3 = \{(2,4), (4,2)\}$,

   (iv) $\mathcal{R}_4 = \{(1,1), (1,3), (2,2), (3,4), (3,3), (4,3), (3,1), (4,4)\}$,

   (v) $\mathcal{R}_5 = \{(1,1), (2,2), (3,3), (4,4)\}$,

   (vi) $\mathcal{R}_6 = \{(1,3), (1,4), (2,3), (2,4), (3,1), (3,4)\}$.

8. Each of the following defines a relation on $\mathbb{Z}$. In each case, determine if the relation is reflexive, symmetric or transitive. **Give your reasons**.

   (i) $x \sim y$ if, and only if, $x + y$ is an odd integer,

   (ii) $x \sim y$ if, and only if, $x + y$ is an even integer,

   (iii) $x \sim y$ if, and only if, xy is an odd integer,

   (iv)★ $x \sim y$ if, and only if, $x + xy$ is an even integer.

9. Which of the following collections of subsets are partitions of $\{1, 2, 3, 4, 5, 6\}$? **Give your reasons**.

   (i) $\{1,2\}, \{2,3,4\}, \{4,5,6\}$,

   (ii) $\{1\}, \{2,3,6\}, \{4\}, \{5\}$,

   (iii) $\{2,4,6\}, \{1,3,5\}$,

   (iv) $\{1,4,5\}, \{2,6\}$.

10. Are the following partitions of $\mathbb{R}$? **Give your reasons**.

   (i) $\{\{x \in \mathbb{R} : x \text{ positive}\}, \{x \in \mathbb{R} : x \text{ negative}\}\}$,

   (ii) $\{T_n : n \in \mathbb{Z}\}$ where $T_n = \{x \in \mathbb{R} : 0 \leq x - n \leq 1\}$,

   (iii) $\{\{x \in \mathbb{R} : x \text{ non-positive}\}, \ \{x \in \mathbb{R} : x \text{ non-negative}\}\}$,

   (iv) $\{U_n : n \in \mathbb{N} \cup \{0\}\}$ where $U_n = \{x \in \mathbb{R} : n \leq |x| < n+1\}$?

11. If $A = \{1, 2, 3, 4, 5\}$ and $\mathcal{R} \subseteq A \times A$ is the equivalence relation that induces the partition $A = \{1, 2\} \cup \{3, 4\} \cup \{5\}$, what is $\mathcal{R}$?

12. If $A = \{1, 2, 3, 4, 5, 6\}$ then

$$\mathcal{R} = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4), (4,5), (5,4), (5,5), (6,6)\}$$

is an equivalence relation.

(i) What are $[1]$, $[2]$,$[3]$,$[4]$, $[5]$ and $[6]$ under this relation?

(ii) What partition of A does $\mathcal{R}$ induce?